

A HILBERT C^* -MODULE ADMITTING NO FRAMES

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ABSTRACT. We show that every infinite-dimensional commutative unital C^* -algebra has a Hilbert C^* -module admitting no frames. In particular, this shows that Kasparov's stabilization theorem for countably generated Hilbert C^* -modules can not be extended to arbitrary Hilbert C^* -modules.

1. INTRODUCTION

Kasparov's celebrated stabilization theorem [4] says that for any C^* -algebra A and any countably generated (right) Hilbert A -module X_A , the direct sum $X_A \oplus H_A$ is isomorphic to H_A as Hilbert A -modules, where H_A denotes the standard Hilbert A -module $\oplus_{j \in \mathbb{N}} A_A$ (see Section 3 below for definition). This theorem plays an important role in Kasparov's KK -theory.

There has been some generalization of Kasparov's stabilization theorem to a larger class of Hilbert C^* -modules [7]. It is natural to ask whether Kasparov's stabilization theorem can be generalized to arbitrary Hilbert A -modules via replacing H_A by $\oplus_{j \in J} A_A$ for some large set J depending on X_A . In other words, given any Hilbert A -module X_A , is $X_A \oplus (\oplus_{j \in J} A_A)$ isomorphic to $\oplus_{j \in J} A_A$ as Hilbert A -modules for some set J ?

An affirmative answer to the above question would imply that X_A is a direct summand of $\oplus_{j \in J} A_A$, i.e, $X_A \oplus Y_A$ is isomorphic to $\oplus_{j \in J} A_A$ for some Hilbert A -module Y_A .

In [2] Frank and Larson generalized the classical frame theory from Hilbert spaces to the setting of Hilbert C^* -modules. Given a unital C^* -algebra A and a Hilbert A -module X_A , a set $\{x_j : j \in J\}$ of elements in X_A is called a *frame* of X_A [2, Definition 2.1] if there is a real constant $C > 0$ such that $\sum_{j \in J} \langle x, x_j \rangle_A \langle x_j, x \rangle_A$ converges in the ultraweak operator topology to some element in the universal enveloping

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von Neumann algebra A^{**} of A [10, page 122] and

$$(1) \quad C \langle x, x \rangle_A \leq \sum_{j \in J} \langle x, x_j \rangle_A \langle x_j, x \rangle_A \leq C^{-1} \langle x, x \rangle_A$$

for every $x \in X_A$. It is called a *standard frame* of X_A if furthermore $\sum_{j \in J} \langle x, x_j \rangle_A \langle x_j, x \rangle_A$ converges in norm for every $x \in X_A$. Frank and Larson showed that a Hilbert A -module X_A has a standard frame if and only if X_A is a direct summand of $\oplus_{j \in J} A_A$ for some set J [2, Example 3.5, Theorems 5.3 and 4.1]. From Kasparov's stabilization theorem they concluded that every countably generated Hilbert A -module has a standard frame. However, the existence of standard frames for general Hilbert A -modules was left open. In fact, even the existence of frames for general Hilbert A -modules is open, as Frank and Larson asked in Problem 8.1 of [2].

The purpose of this note is to show that the answers to these questions are in general negative, even for every infinite-dimensional commutative unital C^* -algebra:

Theorem 1.1. *Let A be a unital commutative C^* -algebra. Then the following are equivalent:*

- (1) *A is finite-dimensional,*
- (2) *for every Hilbert A -module X_A , $X_A \oplus (\oplus_{j \in J} A_A)$ is isomorphic to $\oplus_{j \in J} A_A$ as Hilbert A -modules for some set J ,*
- (3) *for every Hilbert A -module X_A , $X_A \oplus Y_A$ is isomorphic to $\oplus_{j \in J} A_A$ as Hilbert A -modules for some set J and some Hilbert A -module Y_A ,*
- (4) *every Hilbert A -module X_A has a standard frame,*
- (5) *every Hilbert A -module X_A has a frame.*

In Section 2 we establish some result on continuous fields of Hilbert spaces. Theorem 1.1 is proved in Section 3.

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2. CONTINUOUS FIELDS OF HILBERT SPACES

In this section we prove Proposition 2.4.

Lemma 2.1. *There exists an uncountable set S of injective maps $\mathbb{N} \rightarrow \mathbb{N}$ such that for any distinct $f, g \in S$, $f(n) \neq g(n)$ for all but finitely many $n \in \mathbb{N}$, and $f(n) \neq g(m)$ for all $n \neq m$.*

Proof. Take an injective map T from $\cup_{n \in \mathbb{N}} \mathbb{N}^n$ into \mathbb{N} . For each $x : \mathbb{N} \rightarrow \mathbb{N}$ define $f_x : \mathbb{N} \rightarrow \mathbb{N}$ by $f_x(n) = T(x(1), x(2), \dots, x(n))$ for all $n \in \mathbb{N}$. If $x \neq y$, say, $x(m) \neq y(m)$ for some $m \in \mathbb{N}$, then $f_x(n) \neq f_y(n)$

for all $n \geq m$. Now the set $S := \{f_x \in \mathbb{N}^{\mathbb{N}} : x \in \mathbb{N}^{\mathbb{N}}\}$ satisfies the requirement. \square

We refer the reader to [1, Chapter 10] for details on continuous fields of Banach spaces. Let T be a topological space. Recall that a *continuous field of (complex) Banach spaces over T* is a family $(H_t)_{t \in T}$ of complex Banach spaces, with a set $\Gamma \subseteq \prod_{t \in T} H_t$ of sections such that:

- (i) Γ is a linear subspace of $\prod_{t \in T} H_t$,
- (ii) for every $t \in T$, the set of $x(t)$ for $x \in \Gamma$ is dense in H_t ,
- (iii) for every $x \in \Gamma$ the function $t \mapsto \|x(t)\|$ is continuous on T ,
- (iv) for any $x \in \prod_{t \in T} H_t$, if for every $t \in T$ and every $\varepsilon > 0$ there exists an $x' \in \Gamma$ with $\|x'(s) - x(s)\| < \varepsilon$ for all s in some neighborhood of t , then $x \in \Gamma$.

Lemma 2.2. *For each $s \in [0, 1]$ there exists a continuous field of Hilbert spaces $((H_t)_{t \in [0, 1]}, \Gamma)$ over $[0, 1]$ such that H_t is separable for every $t \in [0, 1] \setminus \{s\}$ and H_s is nonseparable.*

Proof. We consider the case $s = 0$. The case $s > 0$ can be dealt with similarly. Let H be an infinite-dimensional separable Hilbert space. Take an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of H . Let S be as in Lemma 2.1. For every $f \in S$ and every $t \in (0, 1]$ set $v_{f,t} \in H$ by

$$v_{f,t} = \cos\left(\frac{1/n - t}{1/n - 1/(n+1)} \cdot \frac{\pi}{2}\right)e_{f(n)} + \sin\left(\frac{1/n - t}{1/n - 1/(n+1)} \cdot \frac{\pi}{2}\right)e_{f(n+1)}$$

for $1/(n+1) \leq t \leq 1/n$. Set H_t to be the closed linear span of $\{v_{f,t} : f \in S\}$ in H for each $t \in (0, 1]$. Let H_0 be a Hilbert space with an orthonormal basis $\{e'_f\}_{f \in S}$ indexed by S . Then H_t is separable for each $t \in (0, 1]$ while H_0 is nonseparable.

For each $f \in S$, consider the section $x_f \in \prod_{t \in [0, 1]} H_t$ defined by $x_f(t) = v_{f,t}$ for $t \in (0, 1]$, and $x_f(0) = e'_f$. Then $x_f(t)$ is a unit vector in H_t for every $t \in [0, 1]$, and the map $t \mapsto x_f(t) \in H$ is continuous on $(0, 1]$. Denote by V the linear span of $\{x_f : f \in S\}$ in $\prod_{t \in [0, 1]} H_t$.

We claim that the function $t \mapsto \|y(t)\|$ is continuous on $[0, 1]$ for every $y \in V$. Let $y \in V$. Say, $y = \sum_{j=1}^m \lambda_j x_{f_j}$ for some pairwise distinct f_1, \dots, f_m in S and some $\lambda_1, \dots, \lambda_m$ in \mathbb{C} . Then the map $t \mapsto y(t) = \sum_{j=1}^m \lambda_j x_{f_j}(t) \in H$ is continuous on $(0, 1]$. Thus the function $t \mapsto \|y(t)\|$ is continuous on $(0, 1]$. When t is small enough, $x_{f_1}(t), \dots, x_{f_m}(t)$ are orthonormal and hence $\|y(t)\| = (\sum_{j=1}^m |\lambda_j|^2)^{1/2}$. Thus the function $t \mapsto \|y(t)\|$ is also continuous at $t = 0$. This proves the claim.

Since V satisfies the conditions (i), (ii), (iii) in the definition of continuous fields of Banach spaces (with Γ replaced by V), by [1, Proposition 10.2.3] one has the continuous field of Hilbert spaces $((H_t)_{t \in [0, 1]}, \Gamma)$

over $[0, 1]$, where Γ is the set of all sections $x \in \prod_{t \in [0, 1]} H_t$ such that for every $t \in [0, 1]$ and every $\varepsilon > 0$ there exists an $x' \in V$ with $\|x'(t') - x(t')\| < \varepsilon$ for all t' in some neighborhood of t . \square

Lemma 2.3. *Let Z be an infinite compact Hausdorff space. Then there exists a real-valued continuous function f on Z such that $f(Z)$ is infinite.*

Proof. Suppose that every real-valued continuous function on Z has finite image. Take a non-constant real-valued continuous function h on Z . Say, $h(Z) = B \cup D$ with both B and D being nonempty finite sets. Then at least one of $h^{-1}(B)$ and $h^{-1}(D)$ is infinite. Say, $h^{-1}(D)$ is infinite. Set $W_1 = h^{-1}(B)$. Then W_1 and $Z \setminus W_1$ are both nonempty closed and open subsets of Z , and $Z \setminus W_1$ is infinite.

Since every real-valued continuous function g on $Z \setminus W_1$ extends to a real-valued continuous function on Z , g must have finite image. Applying the above argument to $Z \setminus W_1$ we can find $W_2 \subseteq Z \setminus W_1$ such that both W_2 and $Z \setminus (W_1 \cup W_2)$ are nonempty closed and open subsets of $Z \setminus W_1$, and $Z \setminus (W_1 \cup W_2)$ is infinite. Inductively, we find pairwise disjoint nonempty closed and open subsets W_1, W_2, W_3, \dots of Z . Now define f on Z by $f(z) = 1/n$ if $z \in W_n$ for some $n \in \mathbb{N}$ and $f(z) = 0$ if $z \in Z \setminus \bigcup_{n=1}^{\infty} W_n$. Then f is a continuous function on Z and $f(Z)$ is infinite, contradicting our assumption. Therefore there exists a real-valued continuous function on Z with infinite image. \square

Proposition 2.4. *Let Z be an infinite compact Hausdorff space. Then there exist a continuous field of Hilbert spaces $((H_z)_{z \in Z}, \Gamma)$ over Z , a countable subset $W \subseteq Z$, and a point $z_\infty \in \overline{W} \setminus W$ such that H_z is separable for every $z \in W$ while H_{z_∞} is nonseparable.*

Proof. By Lemma 2.3 we can find a continuous map $f : Z \rightarrow [0, 1]$ such that $f(Z)$ is infinite. Note that $f(Z)$ is a compact metrizable space. Thus we can find a convergent sequence $\{t_n\}_{n \in \mathbb{N}}$ in $f(Z)$ such that its limit, denoted by t_∞ , is not equal to t_n for any $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ take $z_n \in f^{-1}(t_n)$. Set $W = \{z_n : n \in \mathbb{N}\}$. Then W is countable and $f(\overline{W}) = \overline{\{t_n : n \in \mathbb{N}\}} \ni t_\infty$. Take $z_\infty \in \overline{W}$ with $f(z_\infty) = t_\infty$. Then $z_\infty \notin W$.

By Lemma 2.2 we can find a continuous field of Hilbert spaces $((H'_t)_{t \in [0, 1]}, \Gamma')$ over $[0, 1]$ such that H'_t is separable for every $t \in [0, 1] \setminus \{t_\infty\}$ while H'_{t_∞} is nonseparable. Set $H_z = H'_{f(z)}$ for each $z \in Z$. Then H_z is separable for every $z \in W$ while H_{z_∞} is nonseparable. For each $\gamma \in \Gamma'$ set $x_\gamma \in \prod_{z \in Z} H_z$ by $x_\gamma(z) = \gamma(f(z)) \in H'_{f(z)} = H_z$ for all $z \in Z$. Then $V := \{x_\gamma \in \prod_{z \in Z} H_z : \gamma \in \Gamma'\}$ is a linear subspace of $\prod_{z \in Z} H_z$ satisfying the conditions (i), (ii), (iii) in the definition of continuous

fields of Banach spaces (with Γ replaced by V). By [1, Proposition 10.2.3] one has the continuous field of Hilbert spaces $((H_z)_{z \in Z}, \Gamma)$ over Z , where Γ is the set of all sections $x \in \prod_{z \in Z} H_z$ such that for every $z \in Z$ and every $\varepsilon > 0$ there exists an $x' \in V$ with $\|x'(z') - x(z')\| < \varepsilon$ for all z' in some neighborhood of z . \square

3. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1.

Recall that given a C^* -algebra A , a (*right*) *Hilbert A -module* is a right A -module X_A with an A -valued inner product map $\langle \cdot, \cdot \rangle_A : X_A \times X_A \rightarrow A$ such that:

- (i) $\langle \cdot, \cdot \rangle_A$ is \mathbb{C} -linear in the second variable,
- (ii) $\langle x, ya \rangle_A = \langle x, y \rangle_A a$ for all $x, y \in X_A$ and $a \in A$,
- (iii) $\langle y, x \rangle_A = (\langle x, y \rangle_A)^*$ for all $x, y \in X_A$,
- (iv) $\langle x, x \rangle_A \geq 0$ in A for every $x \in X_A$, and $\langle x, x \rangle_A = 0$ only when $x = 0$,
- (v) X_A is complete under the norm $\|x\| := \|\langle x, x \rangle_A\|^{1/2}$.

Two Hilbert A -modules are said to be *isomorphic* if there is an A -module isomorphism between them preserving the A -valued inner products. We refer the reader to [5, 6, 8, 11] for the basics of Hilbert C^* -modules.

We give a characterization of frames avoiding von Neumann algebras.

Proposition 3.1. *Let A be a unital C^* -algebra and let X_A be a Hilbert A -module. Let $\{x_j : j \in J\}$ be a set of elements in X_A . Then $\{x_j : j \in J\}$ is a frame of X_A if and only if there is a real constant $C > 0$ such that*

$$(2) \quad C\varphi(\langle x, x \rangle_A) \leq \sum_{j \in J} \varphi(\langle x, x_j \rangle_A \langle x_j, x \rangle_A) \leq C^{-1}\varphi(\langle x, x \rangle_A)$$

for every $x \in X_A$ and every state φ of A .

Proof. Suppose that $\{x_j : j \in J\}$ is a frame of X_A . Let C be a constant witnessing (1). Then every state φ of A extends uniquely to a normal state of A^{**} , which we still denote by φ . Applying φ to (1) we obtain (2). This proves the “only if” part.

Now suppose that (2) is satisfied for every $x \in X_A$ and every state φ of A . Let $x \in X_A$. Note that $\langle x, x_j \rangle_A \langle x_j, x \rangle_A = (\langle x_j, x \rangle_A)^* \langle x_j, x \rangle_A \geq 0$ for every $j \in J$. For any finite subset F of J , from (2) we get

$$\varphi\left(\sum_{j \in F} \langle x, x_j \rangle_A \langle x_j, x \rangle_A\right) \leq \varphi(C^{-1} \langle x, x \rangle_A)$$

for every state φ of A , and hence

$$\sum_{j \in F} \langle x, x_j \rangle_A \langle x_j, x \rangle_A \leq C^{-1} \langle x, x \rangle_A.$$

Thus the monotone increasing net $\{\sum_{j \in F} \langle x, x_j \rangle_A \langle x_j, x \rangle_A\}_F$, for F being finite subsets of J ordered by inclusion, of self-adjoint elements in A^{**} is bounded above. Represent A^{**} faithful as a von Neumann algebra on some Hilbert space H . Then we may also represent A^{**} naturally as a von Neumann algebra on the Hilbert space $H^\infty = \oplus_{n \in \mathbb{N}} H$. By [3, Lemma 5.1.4] the above net converges in the weak operator topology of $B(H^\infty)$ to some element a of A^{**} . Since the weak operator topology on $B(H^\infty)$ restricts to the ultraweak operator topology on A^{**} , we see that the above net converges to a in the ultraweak operator topology. Then (2) tells us that

$$C\varphi(\langle x, x \rangle_A) \leq \varphi(a) \leq C^{-1}\varphi(\langle x, x \rangle_A)$$

for every normal state φ of A^{**} . Therefore, $C \langle x, x \rangle_A \leq a \leq C^{-1} \langle x, x \rangle_A$ as desired. This finishes the proof of the “if” part. \square

Let $((H_z)_{z \in Z}, \Gamma)$ be a continuous field of Hilbert spaces over a compact Hausdorff space Z . We shall write the inner product on each H_z as linear in the second variable and conjugate-linear in the first variable. By [1, Proposition 10.1.9] Γ is right $C(Z)$ -module under the pointwise multiplication, i.e.,

$$(xa)(z) = x(z)a(z)$$

for all $x \in \Gamma$, $a \in C(Z)$, and $z \in Z$. By [1, 10.7.1] for any $x, y \in \Gamma$, the function $z \mapsto \langle x(z), y(z) \rangle$ is in $C(Z)$. From the conditions (iii) and (iv) in the definition of continuous fields of Banach spaces in Section 2 one sees that Γ is a Banach space under the supremum norm $\|x\| := \sup_{z \in Z} \|x(z)\|$. Therefore Γ is a Hilbert $C(Z)$ -module with the pointwise $C(Z)$ -valued inner product

$$\langle x, y \rangle_{C(Z)}(z) = \langle x(z), y(z) \rangle$$

for all $x, y \in \Gamma$, and $z \in Z$. In fact, up to isomorphism every Hilbert $C(Z)$ -module arises this way [9, Theorem 3.12], though we won't need this fact except in the case Z is finite.

Lemma 3.2. *Let $((H_z)_{z \in Z}, \Gamma)$ be a continuous field of Hilbert spaces over a compact Hausdorff space Z . Suppose that there are a countable subset $W \subseteq Z$ and a point $z_\infty \in \overline{W} \setminus W$ such that H_z is separable for every $z \in W$ while H_{z_∞} is nonseparable. Then Γ as a Hilbert $C(Z)$ -module has no frames.*

Proof. Suppose that $\{x_j : j \in J\}$ is a frame of Γ . By Proposition 3.1 there is a real constant $C > 0$ such that the inequality (2) holds for every $x \in X_A$ and every state φ of A . For each $z \in Z$ denote by φ_z the state of $C(Z)$ given by evaluation at z . For any $z \in Z$ and any vector $w \in H_z$, by [1, Proposition 10.1.10] we can find $x \in \Gamma$ with $x(z) = w$. Taking $\varphi = \varphi_z$ in the inequality (2), we get

$$(3) \quad C\|w\|^2 \leq \sum_{j \in J} |\langle x_j(z), w \rangle|^2 \leq C^{-1}\|w\|^2.$$

For each $z \in Z$ let S_z be an orthonormal basis of H_z . For each $w \in S_z$, from (3) we see that the set $F_w := \{j \in J : \langle x_j(z), w \rangle \neq 0\}$ is countable. Note that the set $F_z := \{j \in J : x_j(z) \neq 0\}$ is exactly $\bigcup_{w \in S_z} F_w$. For each $z \in W$, since H_z is separable, S_z is countable and hence F_z is countable. Then the set $F := \bigcup_{z \in W} F_z$ is countable.

Since H_{z_∞} is nonseparable, we can find a unit vector $w \in H_{z_\infty}$ orthogonal to $x_j(z_\infty)$ for all $j \in F$. If $j \in J \setminus F$, then $x_j(z) = 0$ for all $z \in W$, and hence by the condition (iii) in the definition of continuous fields of Banach spaces in Section 2 we conclude that $x_j(z_\infty) = 0$. Therefore $\langle x_j(z_\infty), w \rangle = 0$ for all $j \in J$, contradicting (3). Thus Γ has no frames. \square

For any C^* -algebra A , A as a right A -module is a Hilbert A -module with the A -valued inner product $\langle a, b \rangle_A = a^*b$ for all $a, b \in A$. Given a family $\{X_j\}_{j \in J}$ of Hilbert A -modules, their *direct sum*, denoted by $\bigoplus_{j \in J} X_j$, consists of $(x_j)_{j \in J}$ in $\prod_{j \in J} X_j$ such that $\sum_{j \in J} \langle x_j, x_j \rangle_A$ converges in norm, and has the A -valued inner product $\langle (x_j)_{j \in J}, (y_j)_{j \in J} \rangle_A := \sum_{j \in J} \langle x_j, y_j \rangle_A$.

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. (1) \Rightarrow (2): Suppose that A is finite-dimensional. Then $A = C(Z)$ for a finite discrete space Z . For each $z \in Z$ denote by p_z the projection in $C(Z)$ with $p_z(z') = \delta_{z,z'}$ for all $z' \in Z$. Let X_A be a Hilbert A -module. For any $z \in Z$ and any $x p_z, y p_z \in X_A p_z$, one has $\langle x p_z, y p_z \rangle_A \in A p_z = \mathbb{C} p_z$. Thus $\langle x p_z, y p_z \rangle_A = \lambda p_z$ for some $\lambda \in \mathbb{C}$. Set $\langle x p_z, y p_z \rangle = \lambda$. Then it is easily checked that $X_A p_z$ is a Hilbert space under this inner product, $((X_A p_z)_{z \in Z}, \prod_{z \in Z} X_A p_z)$ is a continuous field of Hilbert spaces over Z , and X_A is isomorphic to $\prod_{z \in Z} X_A p_z$ as Hilbert A -modules. Take an infinite-dimensional Hilbert space H such that the Hilbert space dimension of H is no less than that of $X_A p_z$ for all $z \in Z$. Then $X_A p_z \oplus H$ is unitary equivalent to H as Hilbert spaces. It is readily checked that $((H)_{z \in Z}, \prod_{z \in Z} H)$ is a continuous field of Hilbert spaces over Z , and $(\prod_{z \in Z} X_A p_z) \oplus (\prod_{z \in Z} H)$ is isomorphic to $\prod_{z \in Z} H$ as Hilbert A -modules. Let J be an orthonormal basis of H . Then

it is easy to see that $\prod_{z \in Z} H$ and $\oplus_{j \in J} A_A$ are isomorphic as Hilbert A -modules. Therefore $X_A \oplus (\oplus_{j \in J} A_A)$ and $\oplus_{j \in J} A_A$ are isomorphic as Hilbert A -modules. This proves (1) \Rightarrow (2).

The implications (2) \Rightarrow (3) and (4) \Rightarrow (5) are trivial.

The implication (3) \Rightarrow (4) was proved in [2, Example 3.5]. For the convenience of the reader, we indicate the proof briefly here. Suppose that X_A and Y_A are Hilbert A -modules and $X_A \oplus Y_A$ is isomorphic to $\oplus_{j \in J} A_A$ for some set J as Hilbert A -modules. We may assume that $X_A \oplus Y_A = \oplus_{j \in J} A_A$. Denote by P the orthogonal projection $\oplus_{j \in J} A_A \rightarrow X_A$ sending $x + y$ to x for all $x \in X_A$ and $y \in Y_A$. For each $s \in J$ denote by e_s the vector in $\oplus_{j \in J} A_A$ with coordinate $1_A \delta_{j,s}$ at each $j \in J$. Set $x_j = P(e_j)$ for each $j \in J$. For any $x \in X_A$, say, $x = \sum_{j \in J} e_j a_j$ with $a_j \in A$ for each $j \in J$, one has

$$\begin{aligned} \langle x, x \rangle_A &= \sum_{j \in J} a_j^* a_j = \sum_{j \in J} \langle x, e_j \rangle_A \langle e_j, x \rangle_A = \sum_{j \in J} \langle Px, e_j \rangle_A \langle e_j, Px \rangle_A \\ &= \sum_{j \in J} \langle x, Pe_j \rangle_A \langle Pe_j, x \rangle_A = \sum_{j \in J} \langle x, x_j \rangle_A \langle x_j, x \rangle_A. \end{aligned}$$

Therefore $\{x_j : j \in J\}$ is a standard frame of X_A . This proves (3) \Rightarrow (4).

The implication (5) \Rightarrow (1) follows from Proposition 2.4 and Lemma 3.2. \square

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